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Inverse Theorems for Multidimensional Bernstein–Durrmeyer Operators in L_p*

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In this paper, we consider the multidimensional Bernstein–Durrmeyer operators in L_p $(1 \leqslant p < \infty)$ on a simplex. We characterize the rate of approximation by means of K-functionals and the smoothness of the functions they approximate.

1. INTRODUCTION

Bernstein–Durrmeyer polynomial operators were introduced by J. L. Durrmeyer [11] for $f \in L_1[0, 1]$. These operators were brought to the attention of the mathematical community through a paper of M. M. Derriennic [5] in 1981. M. Heilmann [13] considered the saturation of these operators in L_p $(1 \leqslant p < \infty)$. Z. Ditzian and K. G. Ivanov [9] and the author [18] gave the inverse results in L_p and $C[0, 1]$. H. Berens and Y. Xu [3] gave the saturation and inverse theorems for a modified form of these operators with Jacobi weights.

In the multidimensional case, the situation is somewhat more difficult. M. M. Derriennic [6] defined the operator and discussed some of its approximation properties. Recently, the author [19] solved the characterization problem for uniform approximation. In this paper, we give the L_p-inverse theorems on a simplex. Proofs will be given only for the two-dimensional simplex, since the extension to more dimensions does not involve additional difficulties.

The one-dimensional Bernstein–Durrmeyer operators are defined as

$$M_n(f, x) = \sum_{k=0}^{n} P_{n,k}(x)(n+1) \int_0^1 P_{n,k}(t) f(t)\, dt, \tag{1.1}$$

where $f \in L_p[0, 1]$ $(1 \leqslant p \leqslant \infty)$, and $P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

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Now let $S=\{(x, y): x+y \leqslant 1, x, y \geqslant 0\}$ be the two-dimensional simplex. For $f \in L_p(S)$, the two-dimensional Bernstein–Durrmeyer operators are given by

$$D_n(f, x, y)=\sum_{k+m \leqslant n} P_{n,k,m}(x, y)(n+2)(n+1) \iint_S P_{n,k,m}(s, t) f(s, t)\, ds\, dt, \tag{1.2}$$

where $P_{n,k,m}(x, y)=\binom{n}{k}\binom{n-k}{m} x^k y^m (1-x-y)^{n-k-m}$.

We recall some notations.

Denote $(\partial/\partial x) f$ $((\partial/\partial y) f)$ as the derivative of the function f with respect to the first (second) variable.

Let

$$W_p=\left\{f \in L_p(S): \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f \in \text{A.C.}_{\text{loc}}, \phi_1(f)_p<\infty\right\}, \tag{1.3}$$

where for $i=1, 2, 3$,

$$\phi_i(f)_p=\|f\|_p+\phi_{0i}(f)_p+\phi_{0i}(f_1)_p+\phi_{0i}(f_2)_p; \tag{1.4}$$

$$f_1(x, y)=f(1-x-y, y), \qquad f_2(x, y)=f(x, 1-x-y); \tag{1.5}$$

$$\phi_{0i}(f)_p=\max\left\{\left\|x\left(\frac{\partial^2}{\partial x^2} f\right)(x, y)\right\|_{L_p(x+y \leqslant c_i)}, \left\|y\left(\frac{\partial^2}{\partial y^2} f\right)(x, y)\right\|_{L_p(x+y \leqslant c_i)}, \left\|(xy)^{1/2}\left(\frac{\partial^2}{\partial x\, \partial y} f\right)(x, y)\right\|_{L_p(x+y \leqslant c_i)}\right\}; \tag{1.6}$$

$$c_1=\tfrac{3}{4}, \qquad c_2=\tfrac{2}{3}, \qquad c_3=\tfrac{11}{12}.$$

For $f \in L_p(S)$, we define the K-functional

$$K(f, t)_p=\inf_{g \in W_p}\{\|f-g\|_p+t\phi_1(g)_p\}. \tag{1.7}$$

We use the symmetric difference $\Delta_{he} f(v)=f(v+he/2)-f(v-he/2)$ and $\Delta_{he}^2 f(v)=\Delta_{he}(\Delta_{he} f)(v)$ for a vector $v, h \in R$, and a fixed unit vector e. Denote $e_1=(1, 0)$, $e_2=(0, 1)$.

In the following sections, we fix some constants $\frac{2}{3}<a<c<b<\frac{3}{4}$.

2. Rate of Approximation in the Optimal Case

First we need some results for the one-dimensional Bernstein–Durrmeyer operators in [9].

In the one-dimensional case, the weighted Sobolev space is defined as

$$W_p' = \{f \in L_p[0,1] : f' \in \mathrm{A.C.}_{\mathrm{loc}},\ \varphi^2 f'' \in L_p[0,1]\}, \tag{2.1}$$

where $\varphi(x) = (x(1-x))^{1/2}$.

The K-functionals for $f \in L_p[0,1]$ are given by

$$K_{2,\varphi}(f, t^2)_p = \inf_{g \in W_p'} \{\|f-g\|_p + t^2 \|\varphi^2 g''\|_p\}, \tag{2.2}$$

$$\bar{K}_{2,\varphi}(f, t^2)_p = \inf_{g' \in \mathrm{A.C.}_{\mathrm{loc}}} \{\|f-g\|_p + t^2 \|\varphi^2 g''\|_p + t^4 \|g''\|_p\}. \tag{2.3}$$

We have [10]

$$\begin{aligned} \bar{K}_{2,\varphi}(f, t^2)_p &\sim K_{2,\varphi}(f, t^2)_p \\ &\leqslant M\omega_\varphi^{*2}(f, t)_p = M\left\{\int_0^t \int_0^1 |\Delta^2_{h\varphi(x)} f(x)|^p \frac{dx\,dh}{t}\right\}^{1/p}, \end{aligned} \tag{2.4}$$

where M is a constant independent of f and t,

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h), \qquad \text{if} \quad x \in [h, 1-h],$$

$$\Delta_h^2 f(x) = 0, \qquad \text{otherwise.}$$

We need the modified Bernstein–Durrmeyer operators [4]

$$M_n^*(f, x) = \sum_{k=0}^n P_{n,k}(x)(n+2)\int_0^1 P_{n+1,k}(t)\, f(t)\, dt. \tag{2.5}$$

For these operators, we have the following results similar to those in [9, 13].

LEMMA 2.1. *For $f \in L_p[0,1]$ and $M_n^*(f, x)$ defined by* (2.5), *we have a constant A independent of f and n such that*

$$\begin{aligned} \|M_n^* f - f\|_p &\leqslant A\bar{K}_{2,\varphi}(f, 1/n)_p \\ &\leqslant AM\left(n^{1/2}\int_0^{n^{-1/2}} \int_0^1 |\Delta^2_{h\varphi(x)} f(x)|^p\, dx\, dh\right)^{1/p}, \end{aligned} \tag{2.6}$$

$$\|M_n^* f - f\|_p \leqslant A(\|f\|_p + \|\varphi^2 f''\|_p)/(n+1), \tag{2.7}$$

if $f \in W_p'$.

In the weighted Sobolev space W_p, we have three different norms $\{\phi_i(\cdot)_p\}_{i=1,2,3}$. However, they are equivalent, which can be seen from the following lemma.

LEMMA 2.2. *For $f \in W_p$, we have*

$$\phi_3(f)_p \leqslant C\phi_2(f)_p, \tag{2.8}$$

where C is a constant independent of f.

Proof. This fact is trivial if we write out the expressions of the derivatives explicitly:

$$\begin{aligned}
&x\left(\frac{\partial^2}{\partial x^2} f_1\right)(x, y) \\
&\qquad = x\left(\frac{\partial^2}{\partial x^2} f\right)(1-x-y, y), \\
&(xy)^{1/2}\left(\frac{\partial^2}{\partial x\,\partial y} f_1\right)(x, y) \\
&\qquad = (xy)^{1/2}\left(\left(\frac{\partial^2}{\partial x^2} f\right)(1-x-y, y) - \left(\frac{\partial^2}{\partial x\,\partial y} f\right)(1-x-y, y)\right), \\
&y\left(\frac{\partial^2}{\partial y^2} f_1\right)(x, y) \\
&\qquad = y\left(\left(\frac{\partial^2}{\partial x^2} f\right)(1-x-y, y) - 2\left(\frac{\partial^2}{\partial x\,\partial y} f\right)(1-x-y, y)\right. \\
&\qquad\qquad \left. + \left(\frac{\partial^2}{\partial y^2} f\right)(1-x-y, y)\right),
\end{aligned}$$

and hence

$$\begin{aligned}
&\left\| x\left(\frac{\partial^2}{\partial x^2} f\right)(x, y)\right\|_{L_p(c_2 \leqslant x+y \leqslant c_3,\, x \geqslant 1/3)} \\
&\qquad = \left(\iint_{\substack{1-c_3 \leqslant z \leqslant 1/3 \\ y+z \leqslant 2/3}} \left|(1-y-z)\, z\left(\frac{\partial^2}{\partial x^2} f\right)(1-y-z, y)/z\right|^p dy\, dz\right)^{1/p} \\
&\qquad \leqslant 12\left(\iint_{y+z \leqslant 2/3} \left|z\left(\frac{\partial^2}{\partial x^2} f_1\right)(z, y)\right|^p dy\, dz\right)^{1/p} \\
&\qquad \leqslant 12\phi_{02}(f_1).
\end{aligned}$$

By linear combinations of the above formulas with bounded coefficients, we can obtain the other estimates similarly.

To estimate the rate of approximation, we also need the following trivial result.

LEMMA 2.3. *There exists a constant M_p depending only on $\{c_i\}_{i=1}^3$ and $p \in [1, \infty)$, such that*

$$P_{n,k,m}(x, y) \leqslant M_p(n+1)^{-p-3} \qquad \text{for} \quad x+y \leqslant \tfrac{2}{3}$$
$$\text{and } (k+m)/n \geqslant a, \text{ or for } x+y \geqslant c \text{ and } (k+m)/(n+1) \leqslant a. \tag{2.9}$$

Now we can give the direct result in the optimal case.

THEOREM 2.4. *For $1 \leqslant p < \infty$, $f \in W_p$, we have*

$$\|D_n f - f\|_p \leqslant M'\phi_1(f)_p/n, \tag{2.10}$$

where M' is a constant independent of f and n.

Proof. Note that

$$D_n f(x, y) = D_n f_1(1-x-y, y).$$

We only prove

$$\|D_n f - f\|_{L_p(x+y \leqslant 2/3)} \leqslant M'(\|f\|_p + \phi_{01}(f)_p)/n. \tag{2.11}$$

We introduce a decomposition method for multidimensional Bernstein-type operators.

First, let us assume $f|_{x+y \geqslant b} = 0$.

Define $f_s(t) = f(s, (1-s)t)$, for $s, t \in [0, 1]$, we have

$$\begin{aligned}
D_n f(x, y) - f(x, y) \\
&= \sum_{k=0}^{n} P_{n,k}(x)(n+2) \int_0^1 P_{n,k}(s) \\
&\quad \times \left(\sum_{m=0}^{n-k} P_{n-k,m}(y/(1-x))(n+1) \right. \\
&\quad \left. \times \int_0^{1-s} \{P_{n-k,m}(t/(1-s))(f(s, t) - f(s, (1-s)\, y/(1-x)))\}\, dt \right) ds \\
&\quad + \sum_{k=0}^{n} P_{n,k}(x)(n+2) \\
&\quad \times \int_0^1 \left\{ P_{n,k}(s) \left(\sum_{m=0}^{n-k} P_{n-k,m}(y/(1-x))(n+1) \right. \right. \\
&\quad \left. \times \int_0^{1-s} P_{n-k,m}(t/(1-s))\, dt \right) \\
&\quad \left. \times (f(s, (1-s)\, y/(1-x)) - f(x, y)) \right\} ds
\end{aligned}$$

$$= \sum_{k=0}^{n} P_{n,k}(x)(n+2) \int_0^1 \{P_{n+1,k}(s)(M_{n-k}(f_s, y/(1-x))$$

$$- f_s(y/(1-x)))\} \, ds + (M_n^*(f(\cdot, (1-\cdot)\, y/(1-x)), x) - f(x, y))$$

$$:= I + J. \tag{2.12}$$

In the following part, we estimate these two terms. To estimate I, we use Hölder's inequality and Fubini's Theorem, and obtain

$$\|I\|^p_{L_p(x+y \leqslant 2/3)}$$

$$\leqslant \iint_{x+y \leqslant 2/3} \left(\sum_{k=0}^{n} P_{n,k}(x)(n+2) \right.$$

$$\left. \times \int_0^1 P_{n+1,k}(s)\, |M_{n-k}(f_s, y/(1-x)) - f_s(y/(1-x))|^p \, ds \right) dx\, dy$$

$$\leqslant \int_0^{2/3} \left\{ \sum_{k=0}^{n} P_{n,k}(x)(1-x)(n+2) \right.$$

$$\left. \times \int_0^1 P_{n+1,k}(s) \left(\int_0^1 |M_{n-k}(f_s, z) - f_s(z)|^p \, dz \right) ds \right\} dx.$$

From [9] or [13], we get

$$\int_0^1 |M_{n-k}(f_s, z) - f_s(z)|^p \, dz$$

$$\leqslant C_p(\|f_s\|_p^p + \|z(1-z) f_s''(z)\|_p^p)(n-k+1)^{-p}, \tag{2.13}$$

where C_p is a constant independent of n, k, and f_s.

Thus, we have

$$\|I\|^p_{L_p(x+y \leqslant 2/3)}$$

$$\leqslant \int_0^{2/3} \left\{ \sum_{k=0}^{n} P_{n,k}(x)(n+2) \int_0^1 P_{n+1,k}(s)\, C_p(n-k+1)^{-p} \right.$$

$$\times \left(\int_0^1 \left(|f(s, (1-s)z)|^p \right. \right.$$

$$\left. \left. \left. + \left| z(1-z)(1-s)^2 \left(\frac{\partial^2}{\partial y^2} f \right) (s, (1-s)z) \right|^p \right) dz \right) ds \right\} dx$$

$$\leqslant C_p \sum_{k=0}^{n} \int_0^{2/3} P_{n,k}(x)(n+2) \int_0^1 P_{n+1,k}(s)(n-k+1)^{-p}$$
$$\times \left(\int_0^{1-s} \left(|f(s, y)|^p \right.\right.$$
$$\left.\left. + \left| y(1-s-y) \left(\frac{\partial^2}{\partial y^2} f \right) (s, y) \right|^p \right) dy \right) (1-s)^{-1}\, ds\, dx$$
$$\leqslant C_p(n+2) \int_0^1 \sum_{k=0}^{n} P_{n,k}(s)(n-k+1)^{-p-1}$$
$$\times \left(\int_0^{1-s} \left(|f(s, y)|^p + \left| y(1-s-y) \left(\frac{\partial^2}{\partial y^2} f \right) (s, y) \right|^p \right) dy \right) ds.$$

Now let us choose, following [8], an integer $m > p+1$ dependent only on p, for which

$$\sum_{k=0}^{n} P_{n,k}(s)(n/(n-k+1))^{-m} \leqslant m!\,(1-s)^{-m}. \tag{2.14}$$

Therefore by the assumption $f|_{s+y \geqslant b} = 0$, we obtain

$$\|I\|^p_{L_p(x+y \leqslant 2/3)}$$
$$\leqslant C_p(n+2) \int_0^1 \left(\sum_{k=0}^{n} P_{n,k}(s)(n/(n-k+1))^{-m} \right)^{(p+1)/m} n^{-p-1}$$
$$\times \int_0^{1-s} \left(|f(s, y)|^p + \left| y(1-s-y) \left(\frac{\partial^2}{\partial y^2} f \right) (s, y) \right|^p \right) dy\, ds$$
$$\leqslant C_p(n+2)\, n^{-p-1}$$
$$\times \int_0^1 \int_0^{1-s} \left(|f(s, y)|^p + \left| y \left(\frac{\partial^2}{\partial y^2} f \right) (s, y) \right|^p \right)$$
$$\times (m!)^{(p+1)/m} (1-s)^{-p-1}\, ds\, dy$$
$$\leqslant C_p(n+2)\, n^{-p-1}(1-b)^{-p-1}(m!)^{(p+1)/m}$$
$$\times \iint_{s+y \leqslant 3/4} \left(|f(s, y)|^p + \left| y \left(\frac{\partial^2}{\partial y^2} f \right) (s, y) \right|^p \right) ds\, dy$$
$$\leqslant C_p' n^{-p} (\|f\|_p + \phi_{01}(f)_p)^p,$$

where the constant C_p' is independent of f and n.

To estimate J, we note that $f|_{x+y\geqslant 3/4}=0$; we have from (2.7)

$$\begin{aligned}
&\|J\|^p_{L_p(x+y\leqslant 2/3)}\\
&\quad\leqslant \int_0^{2/3} dx \int_0^{2/3} |M_n^*(f(\cdot, (1-\cdot)z), x)-f(x,(1-x)z)|^p\, dz\\
&\quad\leqslant (2A)^p \int_0^{2/3} dz\\
&\qquad\times\Bigg\{(n+1)^{-p}\Bigg[\int_0^1 |f(x,(1-x)z)|^p\, dx\\
&\qquad+3^p\int_0^1 \left|x(1-x)\left(\frac{\partial^2}{\partial x^2}f\right)(x,(1-x)z)\right|^p\\
&\qquad+\left|zx(1-x)\left(\frac{\partial^2}{\partial x\,\partial y}f\right)(x,(1-x)z)\right|^p\\
&\qquad+\left|x(1-x)z^2\left(\frac{\partial^2}{\partial y^2}f\right)(x,(1-x)z)\right|^p dx\Bigg]\Bigg\}\\
&\quad\leqslant (2A)^p(n+1)^{-p}\int_0^1\int_0^{1-x} |f(x,y)|^p/(1-x)\, dy\, dx\\
&\qquad+(6A)^p(n+1)^{-p}\int_0^1\int_0^{1-x}\Bigg\{\left|x\left(\frac{\partial^2}{\partial x^2}f\right)(x,y)\right|^p\\
&\qquad+\left|(xy)^{1/2}\left(\frac{\partial^2}{\partial x\,\partial y}f\right)(x,y)\right|^p\\
&\qquad+\left|y\left(\frac{\partial^2}{\partial y^2}f\right)(x,y)\right|^p\Bigg\}\Big/(1-x)\, dy\, dx\\
&\quad\leqslant 4(2A)^p(n+1)^{-p}\|f\|_p^p+12(6A)^p(n+1)^{-p}(\phi_{01}(f)_p)^p.
\end{aligned}$$

We have now proved that for $f\in W_p$ and $f|_{x+y\geqslant b}=0$,

$$\|D_nf-f\|_{L_p(x+y\leqslant 2/3)}\leqslant C_p''(\|f\|_p+\phi_{01}(f)_p)/n, \tag{2.15}$$

holds with a constant C_p'' depending only on p.

For $f\in W_p$, we now choose $\psi\in C^\infty$, $\psi|_{x+y\leqslant c}=1$ and $\psi|_{x+y\geqslant b}=0$, and define $g=\psi f$ and $h=f-g$. We now write

$$\begin{aligned}
&\|D_nf-f\|_{L_p(x+y\leqslant 2/3)}\\
&\quad\leqslant \|D_ng-g\|_{L_p(x+y\leqslant 2/3)}+\|D_nh\|_{L_p(x+y\leqslant 2/3)}
\end{aligned}$$

$$\leqslant C_p''(\|g\|_p+\phi_{01}(g)_p)/n+\Bigg\|\sum_{k+m\leqslant n} P_{n,k,m}(x,y)(n+1)(n+2)$$

$$\times\iint_S P_{n,k,m}(s,t)\,|h(s,t)|\,ds\,dt\Bigg\|_{L_p(x+y\leqslant 2/3)}. \tag{2.16}$$

For the second term, we use Lemma 2.3, and obtain

$$\Bigg\|\sum_{(k+m)/n\geqslant a} P_{n,k,m}(x,y)(n+2)(n+1)$$

$$\times\iint_S P_{n,k,m}(s,t)\,|h(s,t)|\,ds\,dt\Bigg\|_{L_p(x+y\leqslant 2/3)}$$

$$\leqslant (M_p\,n^{-p-3}(n+2)(n+1)(1+\|\psi\|_\infty))$$

$$\times\iint_S\sum_{k+m\leqslant n} P_{n,k,m}(s,t)\,|f(s,t)|\,ds\,dt$$

$$\leqslant 8M_p(1+\|\psi\|_\infty)\,\|f\|_p/n.$$

We also have from Lemma 2.3 and the assumption $(1-\psi)|_{x+y\leqslant c}=0$

$$\Bigg\|\sum_{(k+m)/n<a} P_{n,k,m}(x,y)(n+2)(n+1)$$

$$\times\iint_S P_{n,k,m}(s,t)\,|(1-\psi(s,t))\,f(s,t)|\,ds\,dt\Bigg\|_{L_p(x+y\leqslant 2/3)}$$

$$\leqslant\Bigg\|\sum_{(k+m)/n<a} P_{n,k,m}(x,y)(n+2)(n+1)$$

$$\times\iint_{s+t\geqslant c} M_p n^{-p-3}\,|f(s,t)|\,ds\,dt\Bigg\|_{L_p(x+y\leqslant 2/3)}(1+\|\psi\|_\infty)$$

$$\leqslant M_p(1+\|\psi\|_\infty)\,n^{-p-3}(n+2)(n+1)\,\|f\|_p/n$$

$$\leqslant 8M_p(1+\|\psi\|_\infty)\,\|f\|_p/n.$$

For the first term, we need to estimate $\phi_{01}(g)_p$ in terms of $\phi_{01}(f)_p+\|f\|_p$. We estimate only $\|x((\partial^2/\partial x^2)\,g)(x,y)\|_{L_p(x+y\leqslant 3/4)}$, since the other two terms can be estimated in the same way.

From our condition on ψ, we know that

$$\psi|_{x+y\geqslant b}=0 \qquad\text{and}\qquad \frac{\partial}{\partial x}\psi\,|_{x+y<c}=0.$$

Therefore we have

$$\left\| x \frac{\partial^2}{\partial x^2} g(x, y) \right\|_{L_p(x+y \leqslant 3/4)}$$

$$= \left\| x \left(\frac{\partial^2}{\partial x^2} \psi \right) (x, y) f(x, y) + 2x \left(\frac{\partial}{\partial x} \psi \right) (x, y) \left(\frac{\partial}{\partial x} f \right) (x, y) \right.$$

$$\left. + x\psi(x, y) \left(\frac{\partial^2}{\partial x^2} f \right) (x, y) \right\|_{L_p(x+y \leqslant 3/4)}$$

$$\leqslant \left\| \frac{\partial^2}{\partial x^2} \psi \right\|_\infty \|f\|_p + 2 \left\| \frac{\partial}{\partial x} f \right\|_{L_p(c \leqslant x+y \leqslant b)} \left\| \frac{\partial}{\partial x} \psi \right\|_\infty + \|\psi\|_\infty \phi_{01}(f)_p. \tag{2.17}$$

Now for any $y \in [c, b]$, we define a function

$$h_y(z) = f((3/4 - y)z, y)$$

and write using [15]

$$\|h_y'\|^p_{L_p[0,1]}$$

$$= \int_0^1 \left| (3/4 - y) \left(\frac{\partial}{\partial x} f \right) ((3/4 - y)z, y) \right|^p dz$$

$$= \int_0^{3/4-y} (3/4 - y)^{p-1} \left| \left(\frac{\partial}{\partial x} f \right) (x, y) \right|^p dx$$

$$\leqslant M_p'(\|h_y\|^p_p + \|z(1-z)\, h_y''(z)\|^p_p)$$

$$= M_p' \left\{ \int_0^{3/4-y} |f(x, y)|^p (3/4 - y)^{-1} \, dx \right.$$

$$\left. + \int_0^{3/4-y} \left| x(3/4 - x - y) \left(\frac{\partial^2}{\partial x^2} f \right) (x, y) \right|^p (3/4 - y)^{-1} \, dx \right\}.$$

Hence

$$\left\| \frac{\partial}{\partial x} f \right\|^p_{L_p(c \leqslant x+y \leqslant b)}$$

$$\leqslant (3/4 - b)^{-p} \left\| (3/4 - y) \left(\frac{\partial}{\partial x} f \right) (x, y) \right\|^p_{L_p(x+y \leqslant 3/4)}$$

$$\leqslant (3/4 - b)^{-p} \int_0^{3/4} \left\{ \int_0^{3/4-y} \left| \left(\frac{\partial}{\partial x} f \right) (x, y) \right|^p dx (3/4 - y)^p \right\} dy$$

$$\leqslant (3/4-b)^{-p} \int_0^{3/4} M_p'(3/4-y)$$

$$\times \left\{ \int_0^{3/4-y} |f(x, y)|^p (3/4-y)^{-1}\, dx \right.$$

$$\left. + \int_0^{3/4-y} \left| x\left(\frac{\partial^2}{\partial x^2} f\right)(x, y)\right|^p (3/4-y)^{-1}\, dx \right\} dy$$

$$\leqslant (3/4-b)^{-p} M_p'(\|f\|_p^p + (\phi_{01}(f)_p)^p).$$

Therefore we obtain from (2.17)

$$\left\| x\left(\frac{\partial^2}{\partial x^2} g\right)(x, y)\right\|_{L_p(x+y\leqslant 3/4)} \leqslant M_p''(\|f\|_p + \phi_{01}(f)_p)$$

with a constant M_p'' depending only on p.

We also have

$$\phi_{01}(\psi f)_p \leqslant M_p''(\|f\|_p + \phi_{01}(f)_p). \tag{2.18}$$

Thus, combining all the above estimates with (2.16), we obtain

$$\|D_n f - f\|_{L_p(x+y\leqslant 2/3)}$$
$$\leqslant C_p''(\|\psi\|_\infty \|f\|_p + M_p''(\|f\|_p + \phi_{01}(f)_p))/n + 16M_p(1+\|\psi\|_\infty)\|f\|_p/n,$$

which implies (2.11).

Our proof is complete.

3. Lemmas

To prove our inverse results, we now give some lemmas.

On the two-dimensional simplex, the Bernstein operators are given by

$$B_n(f, x, y) = \sum_{k+m\leqslant n} f(k/n, m/n)\, P_{n,k,m}(x, y). \tag{3.1}$$

Z. Ditzian obtained the following moments of $B_n(f, x, y)$ in [7].

Lemma 3.1. *For $B_n(f(s, t), x, y)$ given by* (3.1), *we have*

$$\begin{aligned} B_n(s, x, y) &= x; \\ B_n(s^2, x, y) &= x^2 + x(1-x)/n; \\ B_n(st, x, y) &= (1-1/n)\,xy; \\ B_n((1-s-t)^2, x, y) &= (1-x-y)^2 + (x+y)(1-x-y)/n. \end{aligned} \tag{3.2}$$

Similar expressions can be given for the second variable.

Let

$$I_{n,k,m}(x, y) = x^{-1}(1-x-y)^{-1}P_{n,k,m}(x, y)(k(k-1)(1-x-y)^2 - 2k(n-k-m)\,x(1-x-y) + (n-k-m)(n-k-m-1)x^2), \tag{3.3}$$

where the first(third) term vanishes if $k=0$ $(k+m=n)$, and

$$J_{n,k,m}(x, y) = (xy)^{-1/2}(1-x-y)^{-1}P_{n,k,m}(x, y)(km(1-x-y)^2 - (ky+mx)(n-k-m)(1-x-y) + (n-k-m)(n-k-m-1)xy), \tag{3.4}$$

for $k+m \leqslant n$.

For the above formulas, we have

LEMMA 3.2. *For $x+y \leqslant \frac{3}{4}$, there hold*

$$\sum_{k+m \leqslant n} |I_{n,k,m}(x, y)| \leqslant 5n \tag{3.5}$$

and

$$\sum_{k+m \leqslant n} |J_{n,k,m}(x, y)| \leqslant 5n. \tag{3.6}$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
&\sum_{k+m \leqslant n} |I_{n,k,m}(x, y)| \\
&\quad \leqslant n^2 x^{-1}(1-x-y)^{-1} \sum_{k+m \leqslant n} \{P_{n,k,m}(x, y)((k/n)^2(1-x-y)^2 \\
&\qquad - 2k(n-k-m)n^{-2}x(1-x-y) + (n-k-m)^2 n^{-2} x^2 \\
&\qquad + k(1-x-y)^2/n^2 + (n-k-m)x^2/n^2)\} \\
&\quad = n^2 x^{-1}(1-x-y)^{-1} B_n(s^2(1-x-y)^2 - 2s(1-s-t)\,x(1-x-y) \\
&\qquad + (1-s-t)^2 x^2 + (s(1-x-y)^2 + (1-s-t)x^2)/n, x, y) \\
&\quad = n^2 x^{-1}(1-x-y)^{-1} B_n(\{(s-x)(1-x-y) + x((1-x-y) \\
&\qquad - (1-s-t))\}^2 + s(1-x-y)^2/n + (1-s-t)\,x^2/n, x, y) \\
&\quad \leqslant n^2 x^{-1}(1-x-y)^{-1}(2(1-x-y)^2 x(1-x)/n + 2x^2(x+y) \\
&\qquad \times (1-x-y)/n + x(1-x-y)(1-y)/n) \\
&\quad \leqslant 5n.
\end{aligned}$$

For $J_{n,k,m}(x, y)$, note that

$$|km(1-x-y)^2-(ky+mx)(n-k-m)(1-x-y)+(n-k-m)^2 xy|$$
$$=|(k(1-x-y)-(n-k-m)x)|\,|(m(1-x-y)-(n-k-m)y)|. \tag{3.7}$$

We have, using Hölder's inequality and Lemma 3.1,

$$\sum_{k+m\leqslant n} |J_{n,k,m}(x, y)|$$
$$\leqslant (xy)^{-1/2}(1-x-y)^{-1}\Bigg\{\sum_{k+m\leqslant n} (n-k-m)\, xyP_{n,k,m}(x, y)$$
$$+\sum_{k+m\leqslant n} [(P_{n,k,m}(x, y))^{1/2}\,|k(1-x-y)-(n-k-m)x|$$
$$\times (P_{n,k,m}(x, y))^{1/2}\,|m(1-x-y)-(n-k-m)\,y|\,]\Bigg\}$$
$$\leqslant n^2(xy)^{-1/2}(1-x-y)^{-1}\{xy(1-x-y)/n+(B_n(((s-x)$$
$$\times (1-x-y)+x((1-x-y)-(1-s-t)))^2, x, y))^{1/2}$$
$$\times (B_n(((t-y)(1-x-y)+y((1-x-y)-(1-s-t)))^2, x, y))^{1/2}\}$$
$$\leqslant 5n.$$

The proof is complete.

LEMMA 3.3. *For* $I_{n,k,m}(x, y)$ *and* $J_{n,k,m}(x, y)$ *given by* (3.3) *and* (3.4), *there hold*

$$\iint_S |I_{n,k,m}(x, y)|\, dx\, dy \leqslant 2/n \tag{3.8}$$

and

$$\iint_S |J_{n,k,m}(x, y)|\, dx\, dy \leqslant 2/n. \tag{3.9}$$

Proof. Denote $P_{n,k,m}(x, y)=0$, if $(k/n, m/n)\notin S$.
By the equality

$$\iint_S P_{n,k,m}(x, y)\, dx\, dy=(n+2)^{-1}(n+1)^{-1},$$

we have

$$\begin{aligned}
&\iint_S |I_{n,k,m}(x, y)|\, dx\, dy \\
&\quad \leqslant \iint_S P_{n,k,m}(x, y)(k^2(1-x-y)/x - 2k(n-k-m) \\
&\qquad + (n-k-m)^2 x/(1-x-y) + k(1-x-y)/x \\
&\qquad + (n-k-m)x/(1-x-y))\, dx\, dy \\
&\quad = \iint_S (P_{n,k-1,m}(x, y)(n-k-m+1)(k+1) - 2k(n-k-m) \\
&\qquad \times P_{n,k,m}(x, y) + P_{n,k+1,m}(x, y)(k+1)(n-k-m+1))\, dx\, dy \\
&\quad = 2(n-m+1)(n+2)^{-1}(n+1)^{-1} \\
&\quad \leqslant 2/n.
\end{aligned}$$

In the same way, by (3.7) and Hölder's inequality, we have

$$\begin{aligned}
&\iint_S |J_{n,k,m}(x, y)|\, dx\, dy \\
&\quad \leqslant \iint_S (xy)^{-1/2}(1-x-y)^{-1} P_{n,k,m}(x, y)\{(n-k-m)xy \\
&\qquad + |k(1-x-y) - (n-k-m)x|\, |m(1-x-y) - (n-k-m)y|\}\, dx\, dy \\
&\quad \leqslant \left(\iint_S (n-k-m)x P_{n,k,m}(x, y)/(1-x-y)\, dx\, dy\right)^{1/2} \\
&\qquad \times \left(\iint_S (n-k-m)\, y P_{n,k,m}(x, y)/(1-x-y)\, dx\, dy\right)^{1/2} \\
&\qquad + \left(\iint_S P_{n,k,m}(x, y)(k(1-x-y)\right. \\
&\qquad \left. - (n-k-m)x)^2 x^{-1}(1-x-y)^{-1}\, dx\, dy\right)^{1/2} \\
&\qquad \times \left(\iint_S P_{n,k,m}(x, y)(m(1-x-y)\right. \\
&\qquad \left. - (n-k-m)y)^2 y^{-1}(1-x-y)^{-1}\, dx\, dy\right)^{1/2} \\
&\quad \leqslant ((k+1)(n+2)^{-1}(n+1)^{-1}(m+1)(n+2)^{-1}(n+1)^{-1})^{1/2} \\
&\qquad + ((n-m)(n+2)^{-1}(n+1)^{-1}(n-k)(n+2)^{-1}(n+1)^{-1})^{1/2} \\
&\quad \leqslant 2/n.
\end{aligned}$$

The proof is complete.

Now let us give the derivatives of the operators. Let

$$F_{n,k,m}=(n+2)(n+1)\iint_S P_{n,k,m}(s,t)\,f(s,t)\,ds\,dt. \tag{3.10}$$

LEMMA 3.4. *For $D_n(f, x)$ given in (1.2), we have*

$$\left(\frac{\partial^2}{\partial x^2}D_nf\right)(x,y)$$

$$=x^{-1}(1-x-y)^{-1}\sum_{k=0}^{n}\sum_{m=0}^{n-k}I_{n,k,m}(x,y)\,F_{n,k,m} \tag{3.11}$$

$$=n(n-1)\sum_{k=2}^{n}\sum_{m=0}^{n-k}P_{n-2,k-2,m}(x,y)(F_{n,k,m}-2F_{n,k-1,m}+F_{n,k-2,m}); \tag{3.12}$$

$$\left(\frac{\partial^2}{\partial x\,\partial y}D_nf\right)(x,y)$$

$$=(xy)^{-1/2}(1-x-y)^{-1}\sum_{k=0}^{n}\sum_{m=0}^{n-k}J_{n,k,m}(x,y)\,F_{n,k,m} \tag{3.13}$$

$$=n(n-1)\sum_{k=1}^{n}\sum_{m=1}^{n-k}P_{n-2,k-1,m-1}(x,y)$$

$$\times(F_{n,k,m}-F_{n,k-1,m}-F_{n,k,m-1}+F_{n,k-1,m-1}). \tag{3.14}$$

Since the proof of this lemma is the same as that in [7], we omit it here. We can now give the Bernstein-type inequality as follows.

LEMMA 3.5. *For $1\leqslant p<\infty$, $f\in L_p(S)$, we have*

$$\phi_1(D_nf)_p\leqslant 80n\,\|f\|_p. \tag{3.15}$$

Proof. It is sufficient to estimate $\phi_{01}(D_nf)_p$.
For $1<p<\infty$, let $q=p/(p-1)$; then we have from the above lemmas

$$\left\|x\left(\frac{\partial^2}{\partial x^2}D_nf\right)(x,y)\right\|^p_{L_p(x+y\leqslant 3/4)}$$

$$\leqslant 4^p\iint_{x+y\leqslant 3/4}\left(\sum_{k+m\leqslant n}|I_{n,k,m}(x,y)|\right)^{p/q}$$

$$\times\left(\sum_{k+m\leqslant n}|I_{n,k,m}(x,y)|\,|F_{n,k,m}|^p\right)dx\,dy$$

$$\leqslant 4^p(5n)^{p/q} \sum_{k+m\leqslant n}$$

$$\times \iint_{x+y\leqslant 3/4} |I_{n,k,m}(x, y)|\, dx\, dy (n+2)^p(n+1)^p$$

$$\times \left(\iint_S P_{n,k,m}(s, t)\, ds\, dt\right)^{p/q}$$

$$\times \left(\iint_S P_{n,k,m}(s, t)\, |f(s, t)|^p\, ds\, dt\right)$$

$$\leqslant 4^p(5n)^{p-1}(2/n)(n+2)(n+1)$$

$$\times \iint_S \left(\sum_{k+m\leqslant n} P_{n,k,m}(s, t)\, |f(s, t)|^p\right) ds\, dt$$

$$\leqslant (80)^p n^p\, \|f\|_p^p;$$

$$\left\| (xy)^{1/2} \left(\frac{\partial^2}{\partial x\, \partial y} D_n f\right)(x, y)\right\|^p_{L_p(x+y\leqslant 3/4)}$$

$$\leqslant 4^p \iint_{x+y\leqslant 3/4} \left(\sum_{k+m\leqslant n} |J_{n,k,m}(x, y)|\right)^{p/q}$$

$$\times \left(\sum_{k+m\leqslant n} |J_{n,k,m}(x, y)|\, |F_{n,k,m}|^p\right) dx\, dy$$

$$\leqslant 4^p(5n)^{p/q} \sum_{k+m\leqslant n} \iint_{x+y\leqslant 3/4} |J_{n,k,m}(x, y)|\, dx\, dy (n+2)(n+1)$$

$$\times \iint_S P_{n,k,m}(s, t)\, |f(s, t)|^p\, ds\, dt$$

$$\leqslant (80)^p n^p\, \|f\|_p^p.$$

For $p = 1$, the proof is easier, since we need only use Lemma 3.3 and 3.4, and hence we omit it here.

The estimate of $\| y((\partial^2/\partial y^2)\, D_n f)(x, y)\|_{L_p(x+y\leqslant 3/4)}$ can be given in the same way, and our proof of the Bernstein-type inequality is complete.

LEMMA 3.6. *For* $1 \leqslant p < \infty$, $D_n(f, x, y)$ *given in* (1.2) *and* $f \in W_p$, *we have*

$$\phi_1(D_n f)_p \leqslant L\phi_1(f)_p, \tag{3.16}$$

where L *is a constant independent of* f *and* n.

Proof. By Lemma 2.2, we need only prove

$$\phi_2(D_n f)_p \leqslant L\phi_1(f)_p,$$

with a constant L independent of f and n.

Following the fact that

$$(D_n f)_1(x, y) = (D_n(f_1))(x, y),$$

we need only prove

$$\phi_{02}(D_n f)_p \leqslant L(\phi_{01}(f)_p + \|f\|_p). \tag{3.17}$$

First let us assume $f|_{x+y\geqslant b} = 0$. We note that

$$P_{n,k,m}(x, y) - P_{n,k-1,m}(x, y) = -\left(\frac{\partial}{\partial x} P_{n+1,k,m}\right)(x, y)/(n+1), \qquad \text{for} \quad k \geqslant 1,$$

and

$$P_{n,k,m}(x, y) - 2P_{n,k-1,m}(x, y) + P_{n,k-2,m}(x, y) = ((n+2)(n+1))^{-1}\left(\frac{\partial^2}{\partial x^2} P_{n+2,k,m}\right)(x, y), \qquad \text{for} \quad k \geqslant 2, \tag{3.18}$$

and use (3.12) and Hölder's inequality to write for $1 < p < \infty$

$$\begin{aligned}
&\left\| x\left(\frac{\partial^2}{\partial x^2} D_n f\right)(x, y)\right\|^p_{L_p(x+y\leqslant 2/3)} \\
&= \iint_{x+y\leqslant 2/3} \Bigg| n(n-1) \sum_{k\geqslant 2} x P_{n-2,k-2,m}(x, y)(n+2)(n+1) \\
&\quad \times \iint_S (f(s, t)(P_{n,k,m}(s, t) - 2P_{n,k-1,m}(s, t) \\
&\quad + P_{n,k-2,m}(s, t)))\, ds\, dt \Bigg|^p dx\, dy \\
&= \iint_{x+y\leqslant 2/3} \Bigg| n \sum_{k\geqslant 2} (k-1)\, P_{n-1,k-1,m}(x, y) \\
&\quad \times \iint_S f(s, t)\left(\frac{\partial^2}{\partial s^2} P_{n+2,k,m}\right)(s, t)\, ds\, dt \Bigg|^p dx\, dy
\end{aligned}$$

$$= \iint_{x+y \leqslant 2/3} \Bigg| n \sum_{k \geqslant 2} (k-1) P_{n-1,k-1,m}(x, y)$$

$$\times \iint_S P_{n+2,k,m}(s, t) \frac{\partial^2}{\partial s^2} f(s, t)\, ds\, dt \Bigg|^p dx\, dy$$

$$= \iint_{x+y \leqslant 2/3} \Bigg| n \sum_{k \geqslant 2} (k-1) P_{n-1,k-1,m}(x, y)(n+2)/k$$

$$\times \iint_S P_{n+1,k-1,m}(s, t)\, s \left(\frac{\partial^2}{\partial s^2} f\right)(s, t)\, ds\, dt \Bigg|^p dx\, dy$$

$$\leqslant n^p (n+2)^p \iint_{x+y \leqslant 2/3} \Bigg\{ \sum_{k \geqslant 2} P_{n-1,k-1,m}(x, y)((n+2)(n+3))^{-(p-1)}$$

$$\times \iint_S P_{n+1,k-1,m}(s, t) \left| s \left(\frac{\partial^2}{\partial s^2} f\right)(s, t)\right|^p ds\, dt \Bigg\} dx\, dy$$

$$\leqslant n(n+2)(n(n+1))^{-1} \iint_S \left| s \left(\frac{\partial^2}{\partial s^2} f\right)(s, t)\right|^p ds\, dt$$

$$\leqslant 2\phi_{01}(f)_p. \tag{3.19}$$

Now for any $f \in W_p$, we choose the same function ψ as in the proof of Theorem 2.4. Then we have by (2.18) and (3.19)

$$\left\| x \left(\frac{\partial^2}{\partial x^2} D_n(\psi f)\right)(x, y) \right\|_{L_p(x+y \leqslant 2/3)} \leqslant 2\phi_{01}(\psi f)_p$$

$$\leqslant 2M_p''(\|f\|_p + \phi_{01}(f)_p),$$

where M_p'' is a constant independent of f and n.

Let $F'_{n,k,m} = (n+2)(n+1) \iint_S P_{n,k,m}(s, t)(1-\psi(s, t)) f(s, t)\, ds\, dt$.

Using Lemma 2.3 and (3.12), we also have

$$\left\| x \left(\frac{\partial^2}{\partial x^2} (D_n(f-\psi f))\right)(x, y) \right\|^p_{L_p(x+y \leqslant 2/3)}$$

$$= \iint_{x+y \leqslant 2/3} \Bigg| n \sum_{k \geqslant 2} P_{n-1,k-1,m}(x, y)(k-1)$$

$$\times (F'_{n,k,m} - 2F'_{n,k-1,m} + F'_{n,k-2,m}) \Bigg|^p dx\ dy$$

$$\leqslant \iint_{x+y\leqslant 2/3} \left\{ n^2 \sum_{\substack{(k+m-1)/(n-1)\geqslant a\\ k\geqslant 2}} M_p n^{-p-3} \right.$$

$$\times (|F'_{n,k,m}| + 2\,|F'_{n,k-1,m}| + |F'_{n,k-2,m}|)$$

$$+ n^2 \sum_{\substack{(k+m-1)/(n-1)<a\\ k\geqslant 2}} P_{n-1,k-1,m}(x, y)(n+2)(n+1)$$

$$\times \iint_{s+t\geqslant c} |f(s, t)|\, |1-\psi(s, t)|\, (P_{n,k,m}(s, t)$$

$$\left. + 2P_{n,k-1,m}(s, t) + P_{n,k-2,m}(s, t))\, ds\, dt \right\}^p dx\, dy$$

$$\leqslant \iint_{x+y\leqslant 2/3} \left\{ M_p n^{-p-1}(n+2)(n+1)(1+\|\psi\|_\infty) \iint_S |f(s, t)|\, ds\, dt \right.$$

$$\left. + 4M_p n^{-p-3} n^2 (n+2)(n+1)(1+\|\psi\|_\infty) \iint_S |f(s, t)|\, ds\, dt \right\}^p dx\, dy$$

$$\leqslant (64 M_p (1+\|\psi\|_\infty)\, \|f\|_p)^p.$$

Therefore we have proved for $f \in W_p$, $1 < p < \infty$,

$$\left\| x\left(\frac{\partial^2}{\partial x^2} D_n f\right)(x, y)\right\|_{L_p(x+y\leqslant 2/3)} \leqslant (2M_p'' + 64M_p(1+\|\psi\|_\infty))(\|f\|_p + \phi_{01}(f)_p).$$

The case $p = 1$ is easier to prove with the same method, and we omit it. In the same way, by Lemma 3.4 we can write

$$\left\| (xy)^{1/2}\left(\frac{\partial^2}{\partial x\, \partial y} D_n f\right)(x, y)\right\|_{L_p(x+y\leqslant 2/3)} \leqslant L(\|f\|_p + \phi_{01}(f)_p)$$

and

$$\left\| y\left(\frac{\partial^2}{\partial y^2} D_n f\right)(x, y)\right\|_{L_p(x+y\leqslant 2/3)} \leqslant L(\|f\|_p + \phi_{01}(f)_p).$$

The proof is now complete.

4. Main Results

Using all the above lemmas, we can now state and prove our inverse theorems. Here the decomposition method which the author introduced previously [20] is crucial.

THEOREM 1. *For* $1 \leqslant p < \infty$, $f \in L_p(S)$, and $0 < \alpha < 1$, *the following statements are equivalent*:

$$(1) \qquad \|D_n f - f\|_p = O(n^{-\alpha}); \tag{4.1}$$

$$(2) \qquad K(f, t)_p = O(t^\alpha); \tag{4.2}$$

$$(3)(\text{i}) \qquad \|\Delta^2_{h\sqrt{x}e_1} f(x, y)\|_{L_p(x+y \leqslant b, x \geqslant h^2)} \leqslant M h^{2\alpha}; \tag{4.3}$$

$$\|\Delta^2_{t\sqrt{y}e_2} f(x, y)\|_{L_p(x+y \leqslant b,\, y \geqslant t^2)} \leqslant M t^{2\alpha};$$

$$\|\Delta_{h\sqrt{x}e_1} \Delta_{t\sqrt{y}e_2} f(x, y)\|_{L_p(x+y \leqslant b,\, x \geqslant h^2/4,\, y \geqslant t^2/4)} \leqslant M t^\alpha h^\alpha, \tag{4.4}$$

where M *is a constant independent of* $h, t > 0$.

(ii) *Condition* (3)(i) *is valid for* $f_1(x, y) = f(1 - x - y, y)$;

(iii) *Condition* (3)(i) *is valid for* $f_2(x, y) = f(x, 1 - x - y)$.

Proof. The equivalence of (1) and (2) follows from Theorem 2.4, Lemmas 3.5 and 3.6, and a result of A. Grundmann [12].

Now suppose (2) holds, i.e., $K(f, t)_p \leqslant M t^\alpha$.

Note that for $x \geqslant h^2$,

$$\frac{1}{2} \leqslant \frac{d(x \pm h\sqrt{x})}{dx} = 1 \pm \frac{h x^{-1/2}}{2} \leqslant \frac{3}{2}.$$

We have for any $g \in W_p$, $h < \frac{3}{4} - b < \frac{1}{8}$,

$$\begin{aligned}
&\|\Delta^2_{h\sqrt{x}e_1} f(x, y)\|_{L_p(x+y \leqslant b,\, x \geqslant h^2)} \\
&\quad \leqslant \|f - g\|_p + \|f(x + h\sqrt{x}, y) - g(x + h\sqrt{x}, y)\|_{L_p(E)} \\
&\qquad + \|f(x - h\sqrt{x}, y) - g(x - h\sqrt{x}, y)\|_{L_p(E)} + \|\Delta^2_{h\sqrt{x}e_1} g(x, y)\|_{L_p(E)} \\
&\quad \leqslant 5\,\|f - g\|_p + \left\| \iint_{-h/2}^{h/2} \left\{ x \left(\frac{\partial^2}{\partial x^2} g \right) (x + (u + v)\sqrt{x}, y) \right\} du\, dv \right\|_{L_p(E)}
\end{aligned}$$

Here the domain $E = \{(x, y) : x + y \leqslant b, x \geqslant h^2\}$.

We now use a fact in [1]:

$$\iint_{-h/2}^{h/2} ((y + s + t)(1 - y - s - t))^{-1}\, ds\, dt \leqslant 6h^2 (y(1 - y))^{-1}, \tag{4.5}$$

for $0 < h < \frac{1}{8}$, $y \geqslant h$.

Note that

$$x^{1/2}/(x^{1/2} + u + v) \leqslant 1 + h/(h + u + v), \tag{4.6}$$

for $x \geqslant h^2$, $u, v \in (-h/2, h/2)$.

We have by Hölder's inequality for $1 < p < \infty$

$$\iint_{x+y\leqslant b,\, x\geqslant h^2} \left| \iint_{-h/2}^{h/2} \left\{ x\left(\frac{\partial^2}{\partial x^2} g\right)(x+(u+v)x^{1/2}, y)\right\} du\, dv \right|^p dx\, dy$$

$$\leqslant \iint_{x+y\leqslant b,\, x\geqslant h^2} \left\{ \left(\iint_{-h/2}^{h/2} x(x+(u+v)x^{1/2})^{-1}\, du\, dv \right)^{p-1} \right.$$

$$\times \iint_{-h/2}^{h/2} x(x+(u+v)x^{1/2})^{-1}$$

$$\left. \times \left| (x+(u+v)\,x^{1/2}) \left(\frac{\partial^2}{\partial x^2} g\right)(x+(u+v)x^{1/2}, y) \right|^p du\, dv \right\} dx\, dy$$

$$\leqslant (6h^2/(1-b^{1/2}))^{p-1} \iint_{-h/2}^{h/2} (1+h/(h+u+v))\, du\, dv$$

$$\times \iint_{x+y\leqslant b,\, x\geqslant h^2} \left| (x+(u+v)x^{1/2}) \left(\frac{\partial^2}{\partial x^2} g\right)(x+(u+v)\,x^{1/2}, y) \right|^p dx\, dy$$

$$\leqslant (60h^2)^{p-1} 2(\phi_{01}(g)_p)^p (h^2+9h^2)$$

$$\leqslant (60h^2 \phi_{01}(g)_p)^p.$$

The estimate for $p=1$ can be given in a similar way.

Thus we obtain

$$\|\Delta^2_{h\sqrt{x}e_1} f(x, y)\|_{L_p(x+y\leqslant b,\, x\geqslant h^2)}$$

$$\leqslant 60 \inf_{g\in W_p} \{\|f-g\|_p + h^2\phi_1(g)_p\} \leqslant 60Mh^{2\alpha}.$$

Other conditions in (3) can be obtained similarly, and hence we have shown that (2) implies (3).

Now suppose (3) holds; we want to prove (1). It is sufficient to prove $\|D_n f - f\|_{L_p(x+y\leqslant 2/3)} = O(n^{-\alpha})$. From the proof of Theorem 2.4, we can assume $f|_{x+y\geqslant a} = 0$.

We also have the decomposition formula

$$D_n f - f = I + J,$$

as in (2.12). So we estimate these two terms, respectively.

From [9] and (2.4), we have for $s\in(0, 1)$, $0\leqslant k\leqslant n$,

$$\|M_{n-k}(f_s) - f_s\|^p_{L_p[0,1]}$$

$$\leqslant \bar{M}_p(n-k+1)^{1/2} \int_0^{(n-k+1)^{-1/2}} \int_0^1 |\Delta^2_{h\varphi(z)} f_s(z)|^p\, dz\, dh, \qquad (4.7)$$

where the constant $\bar{M}_p$ is independent of k, n, and s.

For $s \in (0, a]$, $y \in (0, b(1-s)]$ and, $h \leqslant b-a$, we have

$$0 < y^{1/2}/4 \leqslant \varphi(y/(1-s))/2 \leqslant y^{1/2};$$

therefore Lemma 2.2.1 of [10] yields

$$\int_0^{(n-k+1)^{-1/2}} \int_0^1 |\Delta^2_{h\varphi(z)} f_s(z)|^p \, dz \, dh$$

$$\leqslant \int_0^{(n-k+1)^{-1/2}} \int_0^{b(1-s)} |\Delta^2_{2h\varphi(y/(1-s))/2e_2} f(s, y)|^p \, dy \, dh/(1-s)$$

$$\leqslant 4 \int_0^{(n-k+1)^{-1/2}} \int_0^{b(1-s)} |\Delta^2_{2h\sqrt{y}e_2} f(s, y)|^p \, dy \, dh/(1-s).$$

Thus, we have, from the above estimate and the assumption $f|_{x+y \geqslant a} = 0$,

$$\begin{aligned}
&\|I\|^p_{L_p(x+y \leqslant 2/3)} \\
&\quad\leqslant \int_0^{2/3} \Bigg\{ \sum_{k=0}^{n} P_{n,k}(x)(1-x)(n+2) \\
&\qquad\times \int_0^1 P_{n+1,k}(s) \, \|M_{n-k}(f_s) - f_s\|^p_{L_p[0,1]} \, ds \Bigg\} \, dx \\
&\quad\leqslant \int_0^{2/3} \Bigg\{ \sum_{0 \leqslant k/n \leqslant b} P_{n,k}(x)(1-x)(n+2) \int_0^a P_{n+1,k}(s) \, \bar{M}_p (n-k+1)^{1/2} 4^2 \\
&\qquad\times \int_0^{(n-k+1)^{-1/2}} \int_0^{b(1-s)} |\Delta^2_{2h\sqrt{y}e_2} f(s, y)|^p \, dy \, dh \, ds \Bigg\} \, dx \\
&\qquad+ \int_0^{2/3} \Bigg\{ \sum_{k/n > b} P_{n,k}(x)(1-x)(n+2) \int_0^a P_{n+1,k}(s) \, 2^p \, \|f_s\|^p_p \, ds \Bigg\} \, dx \\
&\quad\leqslant \int_0^{2/3} \Bigg\{ \sum_{0 \leqslant k/n \leqslant b} P_{n,k}(x)(1-x)(n+2) \int_0^a P_{n+1,k}(s) \, \bar{M}_p \\
&\qquad\times \left((2nb)^{1/2} \, 4^2 \int_0^{(n/4)^{-1/2}} \int_0^{b(1-s)} |\Delta^2_{2h\sqrt{y}e_2} f(s, y)|^p \, dy \, dh \right) ds \Bigg\} \, dx \\
&\qquad+ \int_0^{2/3} \Bigg\{ \sum_{k/n > b} P_{n,k}(x)(1-x)(n+2) \\
&\qquad\times \int_0^1 P_{n+1,k}(s) \, 2^p \int_0^1 |f(s, (1-s)z)|^p \, dz \, ds \Bigg\} \, dx
\end{aligned}$$

$$\leqslant (n+2)(n+1)^{-1}\bar{M}_p(2nb)^{1/2}4^2 \int_0^{(n/4)^{-1/2}} dh$$

$$\times \iint_{s+y\leqslant b} |\Delta^2_{2h\sqrt{y}e_2} f(s, y)|^p \, ds\, dy + \bar{M}'_p n^{-p-1}(n+2)\, 2^p 4 \, \|f\|_p^p$$

$$\leqslant \bar{M}''_p(n^{-\alpha p} + n^{-\alpha p}),$$

where the constant $\bar{M}''_p$ is independent of n. Here, we have used the fact that for $0 \leqslant x \leqslant \frac{2}{3}$, $k/n > b$,

$$P_{n,k}(x) \leqslant \bar{M}'_p n^{-p-1} \tag{4.8}$$

holds with $\bar{M}'_p$ a constant independent of n, k and x.

For the second term J, we have from (2.6)

$$\|J\|^p_{L_p(x+y\leqslant 2/3)}$$

$$\leqslant \int_0^{2/3} dz \int_0^{2/3} |M_n^*(f(\cdot, (1-\cdot)z), x) - f(x, (1-x)z)|^p \, dx$$

$$\leqslant \int_0^{2/3} dz \left\{ (AM)^p n^{1/2} \int_0^{n^{-1/2}} dh \int_0^1 |\Delta^2_{-h\varphi(x)e_1} f(x, (1-x-h\varphi(x)z) \right.$$

$$+ \Delta^2_{h\varphi(x)ze_2} f(x-h\varphi(x), (1-x)z)$$

$$\left. + 2\Delta_{+h\varphi(x)ze_2}\, \Delta_{-h\varphi(x)}\, e_1 f\,(x-h\varphi(x)/2, (1-x-h\varphi(x)/2)z)|^p \, dx \right\}. \tag{4.9}$$

By the assumption $f|_{x+y\geqslant a} = 0$, we have for $n > (c-a)^{-2}$

$$\int_0^{2/3} dz\, n^{1/2} \int_0^{n^{-1/2}} dh \int_0^1 |\Delta^2_{-h\varphi(x)e_1} f(x, (1-x-h\varphi(x)z)|^p \, dx$$

$$\leqslant n^{1/2} \int_0^{n^{-1/2}} dh \int_0^c dx \int_0^{2/3} |\Delta^2_{-h\varphi(x)e_1} f(x, (1-x-h\varphi(x)z)|^p \, dz$$

$$\leqslant 4n^{1/2} \int_0^{n^{-1/2}} dh \int_0^c dx \int_0^{(1-x-h\varphi(x))2/3} |\Delta^2_{-h\varphi(x)e_1} f(x, y)|^p \, dy$$

$$\leqslant 4n^{1/2} \int_0^{n^{-1/2}} dh \int_0^c dy \int_0^{c-y} |\Delta^2_{-h\varphi(x)e_1} f(x, y)|^p \, dx.$$

Note that for $x \in [0, c-y]$,

$$x^{1/2}/2 \leqslant \varphi(x) \leqslant x^{1/2}.$$

We obtain from Lemma 2.2.1 of [10]

$$\int_0^{2/3} dz\, n^{1/2} \int_0^{n^{-1/2}} dh \int_0^1 |\Delta^2_{-h\varphi(x)e_1} f(x, (1-x-h\varphi(x))z)|^p\, dx$$

$$\leqslant 8n^{1/2} \int_0^{n^{-1/2}} dh \iint_{x+y\leqslant c} \Delta^2_{h\sqrt{x}e_1} f(x, y)|^p\, dx\, dy$$

$$\leqslant 8M^p n^{-p\alpha}.$$

The other two terms in (4.9) can be estimated in a similar way.

Therefore,

$$\|D_n f - f\|_{L_p(x+y\leqslant 2/3)} = O(n^{-\alpha}),$$

and the proof of Theorem 1 is complete.

Remark. Another way of proving Theorem 1 is to give an interpolation theorem which is similar to that in [10].

The inverse theorem for m-dimensional Bernstein–Durrmeyer operators on the simplex is given in the following.

For $m \in N$, let $S = \{(x_i) \in R^m : \sum_{i=1}^m x_i \leqslant 1, x_i \geqslant 0\}$, $e_i = (0, ..., 0, \overset{i\text{th}}{1}, 0, ..., 0)$, $x = (x_1, ..., x_m)$, $k = (k_1, ..., k_m)$, and

$$P_{n,k}(x) = n! \left(\prod_{i=1}^m x_i^{k_i}\right)\left(1 - \sum_{i=1}^m x_i\right)^{n - \sum_{i=1}^m k_i} \times \left(\left(\prod_{i=1}^m (k_i!)\right)\left(n - \sum_{i=1}^m k_i\right)!\right)^{-1}.$$

The m-dimensional Bernstein–Durrmeyer operators are given by

$$D_n(f, x) = \sum_{k_i \geqslant 0,\, \sum k_i \leqslant n} P_{n,k}(x)(n+m)!\,(n!)^{-1} \int_S P_{n,k}(y) f(y)\, dy. \tag{4.10}$$

THEOREM 2. *For* $1 \leqslant p < \infty$, $f \in L_p(S)$, $0 < \alpha < 1$, *the following statements are equivalent*:

(1) $\|D_n f - f\|_p = O(n^{-\alpha})$;

(2)(i) *For* $1 \leqslant i, j \leqslant m$

$$\|\Delta^2_{h\sqrt{x_i}e_i} f(x)\|_{L_p(\sum_{l=1}^m x_l \leqslant 1-(2m)^{-1}, x_i \geqslant h^2)} = O(h^{2\alpha});$$

$$\|\Delta_{h\sqrt{x_i}e_i} \Delta_{t\sqrt{x_j}e_j} f(x)\|_{L_p(\sum_{l=1}^m x_l \leqslant 1-(2m)^{-1}, x_i \geqslant h^2/4, x_j \geqslant t^2/4)} = O(h^\alpha t^\alpha);$$

(ii) *For* $1 \leqslant j \leqslant m$, (2)(i) *is valid for* f_j *which is defined by*

$$f_j(x) = f\left(x - x_j e_j + \left(1 - \sum_{l=1}^{m} x_l\right) e_j\right).$$

Thus for multidimensional Bernstein–Durrmeyer operators, we have solved the characterization problem in the non-optimal cases in L_p $(1 \leqslant p < \infty)$ and $C(S)$. The saturation conditions, however, still remain open.

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